On-line technical appendix for "Long Memory Regressors and Predictive Testing: A Two-stage Rebalancing Approach"

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Abstract

This is the on-line technical appendix for our paper entitled "Long Memory Regressors and Predictive Testing: A Two-stage Rebalancing Approach," on second revise and resubmit at *Econometric Reviews*. Included are more detailed proofs of the main theorems and proofs of the lemmas and corollaries, both of which were omitted in the paper itself in order to save space.

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A Appendix

A brief note on references to equations and theorems

Please note that references to equations, lemmas, theorems, and corollaries that are not prefaced by an "A" are references to the body of the main paper. By contrast, references prefaced by an "A" are references to equations or lemmas in the on-line appendix itself. Please also note that the equation numbers in the shortened appendix of the main paper do not match those found here in the on-line appendix.

1.A Lemmas

Lemma A.1. *Let $u_{2,t}$ satisfy* (3) *and* (4) *and define $\bar{\delta} > 0$, and let $|\delta_T| < \bar{\delta}$. Defining $\tilde{u}^{(i)}_{2,t-i}(\delta^x_T) = (1-L)^{\delta^x_T}\tilde{u}^{(i)}_{2,t-i}$, where $\tilde{u}^{(i)}_{2,t-i}$ is defined analogously to* (6)*, we have:*

$$\max_{t\leq T} E(\tilde{u}^{(1)}_{2,t})^2 < \|\Sigma\| \left(\sum_{k=1}^{\infty}\|c_{2k}\|\right)^2 \left(\sum_{v=1}^{\infty}\frac{1}{v^2}\right) < \infty, \tag{A.1}$$

$$\max_{t\leq T} E(\tilde{u}^{(i)}_{2,t})^2 = O\left(\ln(T)^{2(i-1)}\right), \quad \textit{for } i = 2,3 \tag{A.2}$$

$$\max_{t\leq T} E\left[\sup_{|\delta^x_T|<\bar{\delta}}\left(|\tilde{u}^{(i)}_{2,t}(\delta^x_T)|\right)^2\right] = O\left(\left(\ln(T)^{(i-1)}T^{\bar{\delta}}\right)^2\right) \quad \textit{for } i = 2,3. \tag{A.3}$$

Lemma A.2. *Using the same definitions in the statement of Lemma A.1 and defining $\tilde{u}^{(i)}_{1,t}$ analogously to $\tilde{u}^{(i)}_{2,t}$, and defining $\tilde{\varepsilon}^{(i)}_{1,t-i} = ln(1-L)^{(i)}\left(\varepsilon_{1,t}1_{\{t>0\}}\right)$ as a special case, the following results apply*

a) $$T^{-1/2}\sum_{t=1}^{T-1}\underline{\tilde{u}}^{(i)}_{2,t-i}\varepsilon_{1,t+1} = O_p\left(\ln(T)^{(i-1)}\right), \quad \textit{for } i = 1,2,3 \tag{A.4}$$

b) $$T^{-1}\sum_{t=1}^{T-1}(\underline{\tilde{u}}^{(i)}_{2,t})^2 = O_p\left(\ln(T)^{2(i-1)}\right), \quad \textit{for } i = 1,2,3 \tag{A.5}$$

c) $$\sup_{|\delta^x_T|<\bar{\delta}} T^{-1}\sum_{t=1}^{T-1}(\tilde{u}^{(i)}_{2,t}(\delta^x_T))^2 = O_p\left(\ln(T)^{2(i-1)}T^{2\bar{\delta}}\right), \quad \textit{for } i = 2,3 \tag{A.6}$$

d) $$T^{-1}\sum_{t=1}^{T-1}\underline{\tilde{u}}^{(i)}_{k,t-i}\underline{\tilde{u}}^{(j)}_{l,t+h-j} \to_p \bar{\gamma}_{\tilde{u}^{(i)}_k,\tilde{u}^{(j)}_l}(h+i-j),\ |\bar{\gamma}_{\tilde{u}^{(i)}_k,\tilde{u}^{(j)}_l}(h+i-j)| < \infty,\ \textit{for} \tag{A.7}$$

$h,i,j \in \{0,1\}, k,l \in \{1,2\}.$

Remark A.3. *Since $\varepsilon_{1,t}$ can be expressed as a special case of either $u_{1,t}$ or $u_{2,t}$, analogous results hold for $\tilde{\varepsilon}^{(i)}_{1,t-i}$ and $\tilde{\varepsilon}^{(i)}_{1,t-i}(\delta^y_T)$. Likewise, rates on sums of relevant cross products are implied by the rates on the sum of squares by simple application of the Cauchy-Schwarz and Hölder's inequalities.*

1.B Proofs

Proof of Lemma A.1

(A.1) follows by (3), (4) and the series expansion $\ln(x) = \sum_{j=1}^{\infty}(-1)^{j-1}\frac{(x-1)^j}{j}$:

$$\tilde{u}_{2,t-1}^{(1)} = \ln(1-L)u_{2,t}1_{\{t>0\}} = -\sum_{j=1}^{t-1}\frac{1}{j}L^j u_{2,t} = -\sum_{k=0}^{\infty}C_{2k}\sum_{r=k+1}^{k+t-1}\left(\frac{1}{r-k}\right)\varepsilon_{t-r}, \quad \text{and}$$

$$\max_{t\leq T}E(\tilde{u}_{2,t-1}^{(1)})^2 = \max_{t\leq T}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{r=k+1}^{k+t-1}\sum_{s=j+1}^{j+t-1}\left(\frac{1}{r-k}\right)\left(\frac{1}{s-j}\right)C_{2j}E[\varepsilon_{t-r}\varepsilon_{t-s}']C_{2k}' \tag{A.8}$$
$$\leq \|\Sigma\|\left(\sum_{k=0}^{\infty}\|C_{2k}\|\right)^2\left(\sum_{v=1}^{\infty}\left(\frac{1}{v}\right)^2\right) < \infty.$$

For $i > 1$, since $\tilde{u}_{2,t}^{(i)} = ln(1-L)\tilde{u}_{2,t+1}^{(i-1)} = -\sum_{j=1}^{t}L^j\frac{1}{j}\tilde{u}_{2,t+1}^{(i-1)}$, $\tilde{u}_{2,t}^{(i)} = 0$, $t \leq 0$, and $\sum_{j=1}^{T}\frac{1}{j} = O(\ln(T))$,[18]

$$\max_{t\leq T}(\tilde{u}_{2,t}^{(i)})^2 = \max_{t\leq T}\sum_{j=1}^{t}\sum_{k=1}^{t}\frac{1}{k}\frac{1}{j}E\left|\tilde{u}_{2,t+1-j}^{(i-1)}\tilde{u}_{2,t+1-k}^{(i-1)}\right| \leq \max_{t\leq T}E|\tilde{u}_{2,t}^{(i-1)}|^2\left(\sum_{j=1}^{T}\frac{1}{j}\right)^2 \tag{A.9}$$
$$= \max_{t\leq T}E|\tilde{u}_{2,t}^{(i-1)}|^2 O(\ln(T)^2).$$

By (A.8) and recursive application of (A.9) we obtain $\max_{t\leq T}E(\tilde{u}_{2,t}^{(i)})^2 = O\left(ln(T)^{2(i-1)}\right)$ showing (A.2). Let $\psi_{\delta_{T,j}}$ and $\psi_{\bar{\delta}_j}$, j =0,1,2,... denote the Maclaurin coefficients in the expansion of $(1-L)^{\delta_T}$ and $(1-L)^{\bar{\delta}_T}$, respectively. Noting that $\tilde{u}_{2,t}^{(i)} = 0$, for $t \leq 0$, $|\psi_{\delta_{T,j}}| \leq |\psi_{\bar{\delta}_j}|$, where $\psi_{\bar{\delta}_j}$ is non-random, and $\tilde{u}_{2,t}^{(i)}(\delta_T^x) = \sum_{j=0}^{t-1}\psi_{\delta_{T,j}}\tilde{u}_{2,t-j}^{(i)}$, (A.3) then follows by

$$\max_{t\leq T}E\left[\sup_{|\delta_T^x|<\bar{\delta}}\left(|\tilde{u}_{2,t}^{(i)}(\delta_T^x)|\right)^2\right] = \max_{t\leq T}E\left[\sup_{|\delta_T^x|<\bar{\delta}}\left|\sum_{j=0}^{t-1}\sum_{k=0}^{t-1}\psi_{\delta_{T,j}}\tilde{u}_{2,t-j}^{(i)}\psi_{\delta_{T,k}}\tilde{u}_{2,t-k}^{(i)}\right|\right]$$
$$\leq \max_{t\leq T}\sum_{j=0}^{T-1}\left|\psi_{\bar{\delta}_j}\right|\sum_{k=0}^{T-1}\left|\psi_{\bar{\delta}_k}\right|E\left|\tilde{u}_{2,t-j}^{(i)}\tilde{u}_{2,t-k}^{(i)}\right|$$
$$\leq E|\tilde{u}_{2,t}^{(i)}|^2\left(\sum_{j=0}^{T-1}|\psi_{\bar{\delta},j}|\right)^2 = O\left(\left(\ln(T)^{(i-1)}T^{\bar{\delta}}\right)^2\right),$$

since $\sum_{j=0}^{T-1}\psi_{\bar{\delta},j} \approx \sum_{j=0}^{T-1}j^{\bar{\delta}-1} = O(T^{\bar{\delta}})$ (Gradstein and Ryzhik 1994, eqn. 0.121).

Proof of Lemma A.2

We have $T^{-1}\sum_{t=1}^{T-1}(\underline{\tilde{u}}_{2,t-i}^{(i)})^2 = T^{-1}\sum_{t=1}^{T-1}(\tilde{u}_{2,t-i}^{(i)})^2 - \left(T^{-1}\sum_{t=1}^{T-1}\tilde{u}_{2,t-i}^{(i)}\right)^2$. By Jensen's Inequality

$$E\left[\left(T^{-1}\sum_{t=1}^{T-1}\tilde{u}_{2,t-i}^{(i)}\right)^2\right] \leq T^{-1}\sum_{t=1}^{T-1}E(\tilde{u}_{2,t-i}^{(i)})^2 \leq \max_{t\leq T}E(\tilde{u}_{2,t}^{(i)})^2 \quad . \tag{A.10}$$

[18] See Gradstein and Ryzhik (1994), eqn. 0.131.

Thus $E\left|T^{-1}\sum_{t=1}^{T-1}(\underline{\tilde{u}}_{2,t-i}^{(i)})^2\right| = O\left(ln(T)^{2(i-1)}\right) =$ by (A.2). (A.5) follows by Markov's inequality. For (A.4) write $T^{-1/2}\sum_{t=1}^{T-1}\underline{\tilde{u}}_{2,t-i}^{(i)}\varepsilon_{1,t+1} = T^{-1/2}\sum_{t=1}^{T-1}\tilde{u}_{2,t-i}^{(i)}\varepsilon_{1,t+1} - \left(T^{-1}\sum_{t=1}^{T-1}\tilde{u}_{2,t-i}^{(i)}\right)T^{-1/2}\sum_{t=1}^{T-1}\varepsilon_{1,t+1}$. The second term on the RHS is $O_p(\ln(T)^{(i-1)})$ by (A.5), the Cauchy-Schwarz inequality, and the central limit theorem for martingale difference sequences (MDS). For the first term, since $\tilde{u}_{2,t-i}^{(i)}$ is predetermined, by the Law of Iterative Expectations,

$$\begin{aligned} E\left[T^{-1/2}\sum_{t=1}^{T-1}\tilde{u}_{2,t-i}^{(i)}\varepsilon_{1,t+1}\right]^2 &= T^{-1}\sum_{t=1}^{T-1}\sum_{s=1}^{T-1}E\left[\tilde{u}_{2,t-i}^{(i)}\tilde{u}_{2,s-i}^{(i)}\varepsilon_{1,t+1}\varepsilon_{1,s+1}\right] = T^{-1}\sum_{t=1}^{T-1}E\left[(\tilde{u}_{2,t-i}^{(i)})^2\varepsilon_{1,t+1}^2\right] \\ &\leq \max_{t\leq T}E\left[(\tilde{u}_{2,t-i}^{(i)})^2\right]\Sigma_{11} = O\left(ln(T)^{2(i-1)}\right). \end{aligned}$$

Next, (A.6) follows by similar argument as (A.5) since

$$E\left[\left(\sup_{|\delta_T^x|<\bar{\delta}}T^{-1}\sum_{t=1}^{T-1}\tilde{u}_{2,t}^{(i)}(\delta_T^x)\right)^2\right] \leq \max_{t\leq T}E\left[\sup_{|\delta_T^x|<\bar{\delta}}\left(|\tilde{u}_{2,t}^{(i)}(\delta_T^x)|\right)^2\right] = O\left(\ln(T)^{2(i-1)}T^{2\bar{\delta}}\right). \quad \text{(A.11)}$$

For (A.7), since convergence in probability is implied by MSE convergence, we need only show that

$$lim_{T\to\infty}E\left[T^{-1}\sum_{t=1}^{T-1}\underline{\tilde{u}}_{k,t-i}^{(i)}\underline{\tilde{u}}_{l,t+h-j}^{(j)}\right] = \bar{\gamma}_{\tilde{u}_k^{(i)},\tilde{u}_l^{(j)}}(h+i-j) \quad \text{and} \quad \lim_{T\to\infty}\text{var}\left(T^{-1}\sum_{t=1}^{T-1}\underline{\tilde{u}}_{k,t-i}^{(i)}\underline{\tilde{u}}_{l,t+h-j}^{(j)}\right) = 0.$$

Below we consider only the case in which $i = j = 1$ and $h = 0$. The cases when i and/or j is zero are similar but simpler and the cases in which $h = 1$ follow by very similar argument. Substituting

$$\tilde{u}_{k,t-1}^{(1)} = \ln(1-L)u_{i,t-1} = -\sum_{r=1}^{t}\frac{1}{r}u_{i,t-1-r} = -\sum_{r=1}^{t}\frac{1}{r}\sum_{p=0}^{\infty}C_{ip}\varepsilon_{t-1-r-p} \quad \text{for } i = j,k$$

and noting that $E\left[\varepsilon_{t-1-r-p}\varepsilon_{t-1-s-q}\right] = \Sigma$ for $r+p = s+q$ and zero otherwise, we obtain

$$\begin{aligned} \lim_{T\to\infty}E\left[T^{-1}\sum_{t=1}^{T-1}\tilde{u}_{k,t-1}^{(1)}\tilde{u}_{l,t-1}^{(1)}\right] &= lim_{T\to\infty}T^{-1}\sum_{t=1}^{T-1}\sum_{r=1}^{t}\sum_{s=1}^{t}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{1}{r}\frac{1}{s}C_{kp}E\left[\varepsilon_{t-1-s-q}\varepsilon_{t-1-r-p}\right]C_{lq}' \\ &= lim_{T\to\infty}T^{-1}\sum_{t=1}^{T-1}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r\in B_{t,p,q}}\left(\frac{1}{r}\right)\left(\frac{1}{r+p-q}\right)C_{kp}\Sigma_{k,l}C_{lq}' \end{aligned}$$

defining $B_{t,p,q} = \{1+max(q-p,1),\ldots,t+min(q-p,0)\}$. This limit is finite since its argument is bounded:

$$\begin{aligned} &\left|T^{-1}\sum_{t=1}^{T-1}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r\in B_{t,p,q}}\left(\frac{1}{r}\right)\left(\frac{1}{r+p-q}\right)C_{kp}\Sigma_{k,l}C_{lq}'\right| \leq \\ &\quad \sum_{p=0}^{\infty}\|C_{kp}\|\Sigma_{k,l}\sum_{q=0}^{\infty}\|C_{lq}\|\sum_{r\in B_{T,p,q}}\left|\left(\frac{1}{r}\right)\left(\frac{1}{r+p-q}\right)\right| \leq \sum_{p=0}^{\infty}\|C_{kp}\|\Sigma_{k,l}\sum_{q=0}^{\infty}\|C_{lq}\|\sum_{r=1}^{\infty}\frac{1}{r^2} < \infty, \end{aligned}$$

which is finite by (3). $E[\tilde{u}^{(i)}_{k,t-i}] = 0$ and by similar argument $T^{-1}\sum_{t=1}^{T-1}\tilde{u}^{(i)}_{k,t-i}, T^{-1}\sum_{t=1}^{T-1}\tilde{u}^{(j)}_{l,t+h-j} \to_p 0$. Therefore $lim_{T\to\infty}E\left[T^{-1}\sum_{t=1}^{T-1}\underline{\tilde{u}}^{(1)}_{k,t-1}\underline{\tilde{u}}^{(1)}_{l,t-1}\right] = lim_{T\to\infty}E\left[\tilde{u}^{(1)}_{k,t-1}\tilde{u}^{(1)}_{l,t-1}\right] = \bar{\gamma}_{\tilde{u}^{(1)}_k,\tilde{u}^{(1)}_l}(0)$ by definition.

Next we turn to the variance. Since $T^{-1}\sum_{t=1}^{T-1}\tilde{u}^{(i)}_{k,t-i}, T^{-1}\sum_{t=1}^{T-1}\tilde{u}^{(j)}_{l,t+h-j} \to_p 0$, the required result follows from (using (A.10) and the Cauchy-Schwarz and Hölder's Inequalities)

$$\begin{aligned} E\left[\left|T^{-1}\sum_{t=1}^{T-1}\tilde{u}^{(1)}_{k,t-1}\tilde{u}^{(1)}_{l,t-1}\right|^2\right] &\leq E\left[\left(T^{-1}\sum_{t=1}^{T-1}(\tilde{u}^{(1)}_{k,t-1})^2\right)\left(T^{-1}\sum_{t=1}^{T-1}(\tilde{u}^{(1)}_{l,t-1})^2\right)\right] \\ &\leq \left\{E\left|T^{-1}\sum_{t=1}^{T-1}(\tilde{u}^{(1)}_{k,t-1})^2\right|^2\right\}^{1/2}\left\{E\left|T^{-1}\sum_{t=1}^{T-1}(\tilde{u}^{(1)}_{l,t-1})^2\right|^2\right\}^{1/2} < \infty. \end{aligned}$$

Proof of Theorem 2.1

Define $\hat{\delta}^x_T = (\hat{d}_x - d_x)$, where $-\hat{\delta}^x_T$ is the integration order of the second-stage regressor. By assumption

$$T^{\alpha_x}\hat{\delta}^x_T = T^{\alpha_x}(\hat{d}_x - d_x) = O_p(1). \tag{A.12}$$

Using demeaned fitted and true models $\underline{y}_{t+1} = \hat{\beta}_1\underline{\hat{u}}_{2,t} + \hat{\varepsilon}_{1,t+1}$ and $\underline{y}_{t+1} = \beta_1\underline{u}_{2,t} + \underline{\varepsilon}_{1,t+1}$,

$$\sqrt{T}(\hat{\beta}_1 - \beta_1) = \left(T^{-1}\sum_{t=1}^{T-1}\underline{\hat{u}}^2_{2,t}\right)^{-1}\left(T^{-1/2}\sum_{t=1}^{T-1}\underline{\hat{u}}_{2,t}\varepsilon_{1,t+1} + \beta_1 T^{-1/2}\sum_{t=1}^{T-1}(\underline{u}_{2,t} - \underline{\hat{u}}_{2,t})\underline{\hat{u}}_{2,t}\right). \tag{A.13}$$

Let $\bar{\delta} > 0$ and let the indicator $I_{\bar{\delta}}$ take the value 1 if $|\hat{\delta}^x_T| < \bar{\delta}$ and zero otherwise. Let $\eta > 0$. Since $\hat{d}_x \to_p d_x$, for large T, $P(I_{\bar{\delta}} = 0) = P(|\hat{\delta}^x_T| > \bar{\delta}) < \eta$. Thus, $I_{\bar{\delta}} \to_p 1$ and

$$\sqrt{T}(\hat{\beta}_1 - \beta_1) = I_{\bar{\delta}}\sqrt{T}(\hat{\beta}_1 - \beta_1) + (1 - I_{\bar{\delta}})\sqrt{T}(\hat{\beta}_1 - \beta_1) = I_{\bar{\delta}}\sqrt{T}(\hat{\beta}_1 - \beta_1) + o_p(1),$$

where the last term is o_p (1) since $(1 - I_{\bar{\delta}})\sqrt{T}(\hat{\beta}_1 - \beta_1) = 0$ when $I_{\bar{\delta}} = 1$, and $P(I_{\bar{\delta}} = 1) \to 1$. Therefore in what follows below we will assume $|\hat{\delta}^x_T| < \bar{\delta}$ without loss of generality.

Next, applying an exact second order Taylor series expansion to the function $(1-L)^{\hat{\delta}^x_T}$ with argument $\hat{\delta}^x_T$ about zero and where δ^*_T lies between 0 and $\hat{\delta}^x_T$ gives

$$\begin{aligned} (1-L)^{\hat{\delta}^x_T} &= 1 + \hat{\delta}^x_T\ln(1-L) + \frac{1}{2}(\hat{\delta}^x_T)^2\ln(1-L)^2(1-L)^{\delta^*_T} \text{ and} && \text{(A.14)} \\ \hat{u}_{2,t} &= (1-L)^{\hat{\delta}^x_T}u_{2,t}1_{\{t>0\}} = u_{2,t}1_{\{t>0\}} + \hat{\delta}^x_T\tilde{u}^{(1)}_{2,t-1} + \frac{1}{2}(\hat{\delta}^x_T)^2\tilde{u}^{(2)}_{2,t-2}(\delta^*_T) && \text{(A.15)} \end{aligned}$$

where $u^{(2)}_{2,t-2}(\delta^*_T)$ and $\tilde{u}^{(1)}_{2,t-1}$ are defined in Lemma A.1.

Next, we turn to the first term in the numerator of $\sqrt{T}(\hat{\beta}_1 - \beta_1)$ in (A.13). Using (A.15) to substitute for $\hat{u}_{2,t}$, we have

$$T^{-1/2}\sum_{t=1}^{T-1}\underline{\hat{u}}_{2,t}\varepsilon_{1,t+1} = T^{-1/2}\sum_{t=1}^{T-1}\underline{u}_{2,t}\varepsilon_{1,t+1} + R_{1,T} \to_d N(0, \gamma_{u_2,u_2}(0)\Sigma_{11}) \tag{A.16}$$

by Davidson (2000, Theorem 6.2.3, p. 124), since $u_{2,t}\varepsilon_{1,t+1}$ is a strictly stationary MDS,[19] and since, by Lemma A.2 (A.4) and (A.6), and the Cauchy-Schwarz and Hölder inequalities,

$$R_{1,T}=\hat{\delta}_T^x T^{-1/2}\sum_{t=1}^{T-1}\underline{\tilde{u}}_{2,t-1}^{(1)}\varepsilon_{1,t+1}+\frac{1}{2}(\hat{\delta}_T^x)^2T^{-1/2}\sum_{t=1}^{T-1}\underline{\tilde{u}}_{2,t-2}^{(2)}(\delta_T^*)\varepsilon_{1,t+1}=o_p(1) \qquad \text{(A.17)}$$

for $\alpha_x>\frac{1}{4}(1+2\bar{\delta})$, again with $\bar{\delta}$ arbitrarily small.

The behavior of the second term in the numerator of $\sqrt{T}(\hat{\beta}_1-\beta_1)$ in (A.13) is given by

$$\begin{aligned}\beta_1T^{-1/2}\sum_{t=1}^{T-1}(\underline{u}_{2,t}-\underline{\hat{u}}_{2,t})\underline{\hat{u}}_{2,t} &= -\beta_1T^{-1/2}\sum_{t=1}^{T-1}\left(\hat{\delta}_T^x\underline{\tilde{u}}_{2,t-1}^{(1)}+\frac{1}{2}(\hat{\delta}_T^x)^2\underline{\tilde{u}}_{2,t-2}^{(2)}(\delta_T^*)\right)\left(\underline{u}_{2,t}+\hat{\delta}_T^x\underline{\tilde{u}}_{2,t-1}^{(1)}\right.\\ &\left.+\frac{1}{2}(\hat{\delta}_T^x)^2\underline{\tilde{u}}_{2,t-2}^{(2)}(\delta_T^*)\right)=-\beta_1T^{-1/2}\hat{\delta}_T^x\sum_{t=1}^{T-1}\underline{\tilde{u}}_{2,t-1}^{(1)}\underline{u}_{2,t}-\beta_1R_{2,T}=\beta_1O_p(T^{1/2-\alpha_x}),\end{aligned}$$

giving the order of magnitude of the contamination term B_T in (8), where $R_{2,T}$ is defined as, $R_{2,T}=T^{-1/2}\Big[(\hat{\delta}_T^x)^2\sum_{t=1}^{T-1}(\underline{\tilde{u}}_{2,t-1}^{(1)})^2+\frac{1}{2}(\hat{\delta}_T^x)^2\sum_{t=1}^{T-1}\underline{\tilde{u}}_{2,t-2}^{(2)}(\delta_T^*)\underline{u}_{2,t}+(\hat{\delta}_T^x)^3\sum_{t=1}^{T-1}\underline{\tilde{u}}_{2,t-2}^{(2)}(\delta_T^*)\underline{\tilde{u}}_{2,t-1}^{(1)}$ $\frac{1}{4}(\hat{\delta}_T^x)^4\sum_{t=1}^{T-1}(\underline{\tilde{u}}_{2,t-2}^{(2)}(\delta_T^*))^2\Big]$. For $\alpha_x>\frac{1}{4}(1+2\bar{\delta})$, and by Lemma A.2, we have $R_{2,T}=o_p(1)$.

For the denominator of $\sqrt{T}(\hat{\beta}_1-\beta_1)$ in equation (A.13) we have

$$T^{-1}\sum_{t=1}^{T-1}\underline{\hat{u}}_{2,t}^2=T^{-1}\sum_{t=1}^{T-1}\underline{u}_{2,t}^2+R_{3,T}\to_p\gamma_{u_2,u_2}(0) \qquad \text{(A.18)}$$

by standard argument, since by Lemma A.2,

$$\begin{aligned}R_{3,T}= &\quad (\hat{\delta}_T^x)^2T^{-1}\sum_{t=1}^{T-1}(\underline{\tilde{u}}_{2,t-1}^{(1)})^2\frac{1}{4}(\hat{\delta}_T^x)^4T^{-1}\sum_{t=1}^{T-1}\left(\underline{\tilde{u}}_{2,t-2}^{(2)}(\delta_T^*)\right)^2+2\hat{\delta}_T^xT^{-1}\sum_{t=1}^{T-1}\underline{\tilde{u}}_{2,t-1}^{(1)}\underline{u}_{2,t}\\ &+\quad (\hat{\delta}_T^x)^2T^{-1}\sum_{t=1}^{T-1}\underline{\tilde{u}}_{2,t-2}^{(2)}(\delta_T^*)\underline{u}_{2,t}(\hat{\delta}_T^x)^3T^{-1}\sum_{t=1}^{T-1}\underline{\tilde{u}}_{2,t-2}^{(2)}(\delta_T^*)\underline{\tilde{u}}_{2,t-1}^{(1)}=O_p(T^{-2\alpha_x})\\ &+\quad +O_p\left(T^{2\bar{\delta}-4\alpha_x}\ln(T)^2\right)O_p(T^{-\alpha_x})O_p\left(T^{\bar{\delta}-2\alpha_x}\ln(T)\right)+O_p\left(T^{\bar{\delta}-3\alpha_x}\ln(T)\right)=o_p(1),\end{aligned}$$

for $\alpha_x>\bar{\delta}$. Combining the above results shows Theorem 2.1.

Proof of Corollary 2.2

Since $\underline{y}_{t+1}=\hat{\beta}_1\underline{\hat{u}}_{2,t}+\hat{\varepsilon}_{1,t+1}$ and by (A.15) we have, $\hat{\varepsilon}_{1,t+1}=\underline{y}_{t+1}-\hat{\beta}_1\underline{\hat{u}}_{2,t}=\varepsilon_{1,t+1}-(\hat{\beta}_1-\beta_1)\underline{u}_{2,t}-\hat{\delta}_T^x\beta_1\underline{\tilde{u}}_{2,t-1}^{(1)}-\frac{1}{2}(\hat{\delta}_T^x)^2\beta_1\underline{\tilde{u}}_{2,t-2}^{(2)}(\delta_T^*)$. Thus $\hat{\sigma}^2=T^{-1}\sum_{t=1}^{T-1}\hat{\varepsilon}_{1,t+1}^2=T^{-1}\sum_{t=1}^{T-1}\varepsilon_{1,t+1}^2+R_{4,T}=\Sigma_{11}+o_p(1)$

[19] Note that $u_{2,t}$ is a pre-determined short-memory linear process and $\varepsilon_{1,t+1}$ is an i.i.d. series so that the asymptotic normality result employed here is quite standard.

since, for $\alpha_x > \frac{\bar{\delta}}{2}$,

$$\begin{aligned}
R_{4,T} &= (\hat{\beta}_1-\beta_1)^2 T^{-1}\sum_{t=1}^{T-1}\underline{u}_{2,t}^2 + \hat{\beta}_1^2(\hat{\delta}_T^x)^2 T^{-1}\sum_{t=1}^{T-1}(\underline{\tilde{u}}_{2,t-1}^{(1)})^2 + \frac{1}{4}\hat{\beta}_1^2(\hat{\delta}_T^x)^4 T^{-1}\sum_{t=1}^{T-1}(\underline{\tilde{u}}_{2,t-2}^{(2)}(\delta_T^*))^2 \\
&- 2(\hat{\beta}_1-\beta_1)T^{-1}\sum_{t=1}^{T-1}\underline{u}_{2,t}\varepsilon_{1,t+1} - 2\hat{\delta}_T^x\hat{\beta}_1 T^{-1}\sum_{t=1}^{T-1}\underline{\tilde{u}}_{2,t-1}^{(1)}\varepsilon_{1,t+1} - (\hat{\delta}_T^x)^2\hat{\beta}_1 T^{-1}\sum_{t=1}^{T-1}\underline{\tilde{u}}_{2,t-2}^{(2)}(\delta_T^*)\varepsilon_{1,t+1} \\
&+ 2\hat{\delta}_T^x(\hat{\beta}_1-\beta_1)\hat{\beta}_1 T^{-1}\sum_{t=1}^{T-1}\underline{u}_{2,t}\underline{\tilde{u}}_{2,t-1}^{(1)} + (\hat{\delta}_T^x)^2\left(\hat{\beta}_1-\beta_1\right)\hat{\beta}_1 T^{-1}\sum_{t=1}^{T-1}\underline{u}_{2,t}\underline{\tilde{u}}_{2,t-2}^{(2)}(\delta_T^*) \\
&+ (\hat{\delta}_T^x)^3\hat{\beta}_1^2 T^{-1}\sum_{t=1}^{T-1}\underline{\tilde{u}}_{2,t-1}^{(1)}\underline{\tilde{u}}_{2,t-2}^{(2)}(\delta_T^*) = o_p(1)
\end{aligned}$$

by (9), (A.12), (A.4) - (A.6), and the Cauchy-Schwarz and Hölder inequalities.

Proof of Corollary 2.3

Result (a) follows from Theorem 2.1 and Corollary 2.2 by standard arguments (note that B_T is not present under the null $H_o: \beta_1 = 0$). For (**b**) note that under $H_A: \beta_1 \neq 0$, $\hat{\beta}_1 - \beta_1 = O_p(T^{-\alpha_x})$ by (9). Therefore

$$\begin{aligned}
T^{-1/2}t &= \hat{\sigma}^{-1}\left(T^{-1}\sum_{t=1}^{T-1}\underline{\hat{u}}_{2,t}^2\right)^{1/2}\hat{\beta}_1 = \hat{\sigma}^{-1}\left(T^{-1}\sum_{t=1}^{T-1}\underline{\hat{u}}_{2,t}^2\right)^{1/2}\beta_1 + \hat{\sigma}^{-1}\left(T^{-1}\sum_{t=1}^{T-1}\underline{\hat{u}}_{2,t}^2\right)^{1/2}(\hat{\beta}_1-\beta_1) \\
&\to {}_p\Sigma_{11}^{-1/2}\gamma_{u_2,u_2}(0)^{1/2}\beta_1
\end{aligned}$$

since the second term is o_p (1) on account of the consistency of $\hat{\beta}_1$ for β_1.

Proof of Theorem 2.4

Define $\hat{\delta}_T^y = \hat{d}_y - d_y$ and let $\bar{\delta} > 0$. By the arguments of Theorem 2.1 assume $\hat{d}_x, \hat{d}_y < \bar{\delta}$ without loss of generality. The denominator of $\hat{\beta}_1$ in (13) is unchanged relative to Theorem 2.1. Using (11) to substitute for y_{t+1}, we note that $T^{(\alpha_y-1)}$ times the numerator is given by

$$T^{(\alpha_y-1)}\sum_{t=1}^{T-1}(1-L)^{\hat{d}_y}y_{t+1}\underline{\hat{u}}_{2,t} = T^{(\alpha_y-1)}\sum_{t=1}^{T-1}(1-L)^{\hat{d}_y-d_y}\left(\varepsilon_{1,t+1}1_{\{t>0\}}\right)\underline{\hat{u}}_{2,t}.$$

Similarly to Theorem 2.1, an exact second order Taylor series expansion of $\hat{\delta}_T^y$, with $0 \leq \delta_T^{*y} \leq \hat{\delta}_T^y$, gives

$$(1-L)^{\hat{\delta}_T^y} = 1 + \hat{\delta}_T^y\ln(1-L) + \frac{1}{2}(\hat{\delta}_T^y)^2\left[\ln(1-L)\right]^2(1-L)^{\delta_T^{*y}}. \tag{A.19}$$

Then, employing definitions analogous to those of Lemma A.1,

$$(1-L)^{\hat{d}_y-d_y}\left(\varepsilon_{1,t+1}1_{\{t>0\}}\right)\underline{\hat{u}}_{2,t} = \left(\varepsilon_{1,t+1} + \hat{\delta}_T^y\tilde{\varepsilon}_{1,t}^{(1)} + \frac{1}{2}(\hat{\delta}_T^y)^2\tilde{\varepsilon}_{1,t-1}^{(2)}(\delta_T^{*y})\right)\underline{\hat{u}}_{2,t}.$$

Therefore, $T^{(\alpha_y-1)}$ times the numerator is given by the three terms:

$$T^{(\alpha_y-1)}\sum_{t=1}^{T-1}[(1-L)^{\hat{d}_y-d_y}\varepsilon_{1,t+1}\underline{\hat{u}}_{2,t}] = T^{(\alpha_y-1)}\sum_{t=1}^{T-1}\varepsilon_{1,t+1}\underline{\hat{u}}_{2,t}+\hat{\delta}_T^y T^{(\alpha_y-1)}\sum_{t=1}^{T-1}\tilde{\varepsilon}_{1,t}^{(1)}\underline{\hat{u}}_{2,t} + \frac{1}{2}(\hat{\delta}_T^y)^2 T^{(\alpha_y-1)}\sum_{t=1}^{T-1}\tilde{\varepsilon}_{1,t-1}^{(2)}(\delta_T^{*y})\underline{\hat{u}}_{2,t}. \quad (A.20)$$

The behavior of the first term in (A.20) is derived in (A.16), from which we can see that

$$T^{(\alpha_y-1)}\sum_{t=1}^{T-1}\varepsilon_{1,t+1}\underline{\hat{u}}_{2,t}\to_d 1_{\{\alpha_y=1/2\}}\gamma_{u_2,u_2}(0)\xi_\beta \quad (A.21)$$

where ξ_β is specified in (16). Substituting (A.15) for $\underline{\hat{u}}_{2,t}$, the second term in (A.20) is given by

$$\hat{\delta}_T^y T^{(\alpha_y-1)}\sum_{t=1}^{T-1}\tilde{\varepsilon}_{1,t}^{(1)}\underline{\hat{u}}_{2,t} = \hat{\delta}_T^y T^{(\alpha_y-1)}\sum_{t=1}^{T-1}\tilde{\varepsilon}_{1,t}^{(1)}\underline{u}_{2,t}+\hat{\delta}_T^y\hat{\delta}_T^x T^{(\alpha_y-1)}\sum_{t=1}^{T-1}\tilde{\varepsilon}_{1,t}^{(1)}\underline{\tilde{u}}_{2,t-1}^{(1)} + \frac{1}{2}\hat{\delta}_T^y(\hat{\delta}_T^x)^2 T^{(\alpha_y-1)}\sum_{t=1}^{T-1}\tilde{\varepsilon}_{1,t}^{(1)}\underline{\tilde{u}}_{2,t-2}^{(2)}(\delta_T^*). \quad (A.22)$$

For the first term, using Lemma A.2 (A.7), $T^{-(1-\alpha_y)}\hat{\delta}_T^y\sum_{t=1}^{T-1}\tilde{\varepsilon}_{1,t}^{(1)}\underline{u}_{2,t} = T^{\alpha_y}\hat{\delta}_T^y T^{-1}\sum_{t=1}^{T-1}\tilde{\varepsilon}_{1,t}^{(1)}\underline{u}_{2,t}\to_d \bar{\gamma}_{u_2,\tilde{\varepsilon}_1^{(1)}}(0)\xi_{\delta^y}$, where the distribution of ξ_{δ^y} is specified in (14) and (16). By (A.5)-(A.6) and the Cauchy-Schwarz and Hölder inequalities, the remaining two terms in (A.22) are $O_p\left(T^{-\alpha_x}\right)$ and $O_p\left(T^{(\bar{\delta}-2\alpha_x)}\ln(T)\right) = o_p(1)$ respectively.

Then the third main term in (A.20) is proportional to:

$$\left(\hat{\delta}_T^y\right)^2 T^{(\alpha_y-1)}\sum_{t=1}^{T-1}\tilde{\varepsilon}_{1,t-1}^{(2)}(\delta_T^{*y})\underline{\hat{u}}_{2,t} = \left(\hat{\delta}_T^y\right)^2 T^{(\alpha_y-1)}\sum_{t=1}^{T-1}\tilde{\varepsilon}_{1,t-1}^{(2)}(\delta_T^{*y})\underline{u}_{2,t}+\left(\hat{\delta}_T^y\right)^2\hat{\delta}_T^x T^{(\alpha_y-1)} \times\sum_{t=1}^{T-1}\tilde{\varepsilon}_{1,t-1}^{(2)}(\delta_T^{*y})\underline{\tilde{u}}_{2,t-1}^{(1)}\frac{1}{2}\left(\hat{\delta}_T^y\right)^2(\hat{\delta}_T^x)^2 T^{(\alpha_y-1)}\sum_{t=1}^{T-1}\tilde{\varepsilon}_{1,t-1}^{(2)}(\delta_T^{*y})\underline{\tilde{u}}_{2,t-2}^{(2)}(\delta_T^*). \quad (A.23)$$

Using (A.5)-(A.6) and the Schwarz and Hölder inequalities, the three terms in (A.23) are respectively $O_p\left(T^{(\bar{\delta}-\alpha_y)}ln(T)\right) = o_p(1)$, $O_p\left(T^{(\bar{\delta}-\alpha_y-\alpha_x)}ln(T)\right) = o_p(1)$, and $O_p\left(T^{(2\bar{\delta}-\alpha_y-2\alpha_x)}ln(T)^2\right) = o_p(1)$.

Proof of Lemma 2.5

Since $(1-L)^d y_{t+1} = (1-L)^{d_y}y_{t+1}+y_t^*$ where $y_t^* = \sum_{j=1}^{\infty}\psi_j(1-L)^{d_y}y_{t+1-j}$ and $\psi_j = \Gamma(j+d_y-d)/\left[\Gamma(d_y-d)\Gamma(j+1)\right]$,

$$2E\left[q_{t+1}(b_1,d,d_x)\right] = E\left[\left(\varepsilon_{1,t+1}+y_t^*+(b_1-\beta_1)(1-L)^{d_x}x_t\right)^2\right] = E\left[\varepsilon_{1,t+1}^2\right] + E\left[\left(y_t^*+(b_1-\beta_1)(1-L)^{d_x}x_t\right)^2\right] > E\left[\varepsilon_{1,t+1}^2\right] = 2E\left[q_{t+1}(\beta_1,d_y,d_x)\right],$$

because both y_t^* and $(b_1-\beta_1)(1-L)^{d_x}x_t$ are predetermined and thus orthogonal to $\varepsilon_{1,t+1}$.

Proof of Theorem 2.6

Define the i^{th} derivative of $\hat{\beta}_1(d_y+\delta_T^y)$ with respect to δ_T^y, expressed as a function of δ_T^y, as

$$\hat{\beta}_1^{(i)}(\delta_T^y) = \left(\sum_{t=1}^{T-1}[\underline{\hat{u}}_{2,t}]^2\right)^{-1}\sum_{t=1}^{T-1}\underline{\hat{u}}_{2,t}\ln(1-L)^{(i)}(1-L)^{d_y+\delta_T^y}y_{t+1}.$$

Let $\bar{i}=\max(i,1)$ and $|\delta_T^y|<\bar{\delta}$. Then using (A.18) and Lemma A.2 we have

$$\begin{aligned}\hat{\beta}_1^{(i)}(\delta_T^y) &= \left(\sum_{t=1}^{T-1}[\underline{\hat{u}}_{2,t}]^2\right)^{-1}\sum_{t=1}^{T-1}\underline{\hat{u}}_{2,t}\ln(1-L)^{(i)}(1-L)^{\delta_T^y}\left(\beta_1u_{2,t}+\varepsilon_{1,t+1}\right) &\text{(A.24)}\\ &= \left(\sum_{t=1}^{T-1}[\underline{\hat{u}}_{2,t}]^2\right)^{-1}\sum_{t=1}^{T-1}\left(\underline{u}_{2,t}+\hat{\delta}_T^x\underline{\tilde{u}}_{2,t-1}^{(1)}+\frac{1}{2}(\hat{\delta}_T^x)^2\underline{\tilde{u}}_{2,t-2}(\delta_T^x)\right)\left(\beta_1\tilde{u}_{2,t-i}^{(i)}(\delta_T^y)+\tilde{\varepsilon}_{1,t+1-i}^{(i)}(\delta_T^y)\right)\\ &= O_p\left(\ln(T)^{(\bar{i}-1)}T^{\bar{\delta}}\right)\end{aligned}$$

Likewise, we define the special case where $\delta_T^y=0$ as $\hat{\beta}_1^{(i)}=\hat{\beta}_1^{(i)}(0)=O_p\left(\ln(T)^{(\bar{i}-1)}\right)$. Additionally, $\hat{\beta}_1^{(0)}(d_y)-\beta_1=O_p(T^{-\alpha_x})$ holds by (9) and defining $\beta_1^{(1)}=\gamma_{u_2,u_2}(0)^{-1}(\beta_1\bar{\gamma}_{u_2,\tilde{u}_2^{(1)}}(-1)+\bar{\gamma}_{u_2,\tilde{\varepsilon}_1^{(1)}}(0))$, it follows from (A.7), (A.18), and (A.24) that $\hat{\beta}_1^{(1)}=\beta_1^{(1)}+o_p(1)$. It will also be useful to define:

$$\begin{aligned}r_{1,t}^{(i)}(\delta_T^y) &= \tilde{\varepsilon}_{1,t+1-i}^{(i)}(\delta_T^y)+\beta_1\underline{\tilde{u}}_{2,t-i}^{(i)}(\delta_T^y)-\hat{\beta}_1^{(i)}(\delta_T^y)\underline{u}_{2,t}, \quad \text{for} \quad i=0,1,2,3 &\text{(A.25)}\\ r_{2,t}^{(i)}(\delta_T^y) &= -\hat{\beta}_1^{(i)}(\delta_T^y)(\underline{\hat{u}}_{2,t}-\underline{u}_{2,t}) \quad \text{for} \quad i=0,1,2,3 &\text{(A.26)}\\ r_{3,t}^{(0)} &= -\left(\hat{\beta}_1^{(0)}-\beta_1\right)\underline{u}_{2,t}. &\text{(A.27)}\end{aligned}$$

For the special case that $\delta_T^y=0$, we define $r_{1,t}^{(i)}=\tilde{\varepsilon}_{1,t+1-i}^{(i)}+\beta_1\underline{\tilde{u}}_{2,t-i}^{(i)}-\hat{\beta}_1^{(i)}\underline{u}_{2,t}$ and $r_{2,t}^{(i)}=\hat{\beta}_1^{(i)}(\underline{\hat{u}}_{2,t}-\underline{u}_{2,t})$.

The following convergence rates are a consequence of Lemma A.2 (and application of a standard LLN to $u_{2,t}$)

$$\begin{aligned}\sum_{t=1}^{T-1}r_{1,t}^{(i)}(\delta_T^y)r_{1,t}^{(j)}(\delta_T^y) &= \sum_{t=1}^{T-1}\Big[\left(\tilde{\varepsilon}_{1,t+1-i}^{(i)}(\delta_T^y)+\beta_1\underline{\tilde{u}}_{2,t-i}^{(i)}(\delta_T^y)-\hat{\beta}_1^{(i)}(\delta_T^y)\underline{u}_{2,t}\right) &\text{(A.28)}\\ &\quad\times\left(\tilde{\varepsilon}_{1,t+1-j}^{(j)}(\delta_T^y)+\beta_1\underline{\tilde{u}}_{2,t-j}^{(j)}(\delta_T^y)-\hat{\beta}_1^{(j)}(\delta_T^y)\underline{u}_{2,t}\right)\Big]=O_p\left(T^{1+2\bar{\delta}}\ln(T)^{(\bar{i}+\bar{j}-2)}\right)\\ \sum_{t=1}^{T-1}r_{2,t}^{(i)}(\delta_T^y)r_{2,t}^{(j)}(\delta_T^y) &= -\sum_{t=1}^{T-1}\hat{\beta}_1^{(i)}(\delta_T^y)\hat{\beta}_1^{(j)}(\delta_T^y)(\hat{\delta}_T^x\underline{\tilde{u}}_{2,t-1}^{(1)}+\frac{1}{2}(\hat{\delta}_T^x)^2\underline{\tilde{u}}_{2,t-2}^{(2)}(\delta_T^*))^2 &\text{(A.29)}\\ &= O_p\left(T^{1+2\bar{\delta}-2\alpha_x}\ln(T)^{\bar{i}+\bar{j}-2)}\right)\\ \sum_{t=1}^{T-1}r_{3,t}^{(0)}r_{3,t}^{(0)} &= \left(\hat{\beta}_1^{(0)}-\beta_1\right)^2\sum_{t=1}^{T-1}\underline{u}_{2,t}^2=O_p(T^{1-2\alpha_x}). &\text{(A.30)}\end{aligned}$$

Plugging $\bar{\delta}=0$ into (A.28) and (A.29) gives the rates for $\sum_{t=1}^{T-1}r_{k,t}^{(i)}r_{k,t}^{(j)}$ for $k=1,2$.

Noting that $\varepsilon_{1,t+1}$ is a MDS and $u_{2,t}$, is predetermined so that $T^{-1/2}\sum_{t=1}^{T-1}\varepsilon_{1,t+1}u_{2,t}$ converges weakly and employing (A.15) and the results of Lemmas A.2 we also have, for any $\bar{\delta}>0$,

$$\sum_{t=1}^{T-1}\varepsilon_{1,t+1}r_{1,t}^{(i)} = \sum_{t=1}^{T-1}\varepsilon_{1,t+1}\left(\tilde{\varepsilon}_{1,t+1-i}^{(i)}+\beta_1^{(0)}\tilde{\underline{u}}_{2,t-i}^{(i)}-\hat{\beta}_1^{(i)}\underline{u}_{2,t}\right)=O_p\left(T^{1/2}\ln(T)^{(\bar{i}-1)}\right) \quad \text{(A.31)}$$

$$\sum_{t=1}^{T-1}\varepsilon_{1,t+1}r_{2,t}^{(i)} = -\hat{\beta}_1^{(i)}\hat{\delta}_T^x\left[\sum_{t=1}^{T-1}\varepsilon_{1,t+1}\tilde{\underline{u}}_{2,t-1}^{(1)}+1/2\hat{\delta}_T^x\sum_{t=1}^{T-1}\varepsilon_{1,t+1}\tilde{\underline{u}}_{2,t-2}^{(2)}(\delta_T^*)\right] \quad \text{(A.32)}$$

$$= O_p\left(\ln(T)^{(\bar{i}-1)}T^{1/2-\alpha_x}\right)+O_p\left(ln(T)^{\bar{i}}T^{1+\bar{\delta}-2\alpha_x}\right)=O_p\left(ln(T)^{\bar{i}}T^{1+\bar{\delta}-2\alpha_x}\right)$$

$$\sum_{t=1}^{T-1}\varepsilon_{1,t+1}r_{3,t}^{(0)} = -\left(\hat{\beta}_1^{(0)}-\beta_1\right)\sum_{t=1}^{T-1}\varepsilon_{1,t+1}\underline{u}_{2,t}=O_p(T^{1/2-\alpha_x}). \quad \text{(A.33)}$$

We next define $e_{t+1}(b,d)=(1-L)^d y_{t+1}-b(1-L)^{\hat{d}_x}x_t$. Denoting, $e_{t+1}^{(i)}$ as the i^{th} partial derivative of e_{t+1} with respect to d (evaluated at $d=d_y+\delta_T^y$), by recursive calculation, we obtain

$$e_{t+1}^{(i)}\left(\hat{\beta}_1^{(i)}(\delta_T^y),d_y+\delta_T^y\right) = \ln(1-L)^{(i)}(1-L)^{d_y+\delta_T^y}y_{t+1}-\hat{\beta}_1^{(i)}(\delta_T^y)\underline{(1-L)^{\hat{d}_x}x_t} \quad \text{(A.34)}$$

$$= \left[\tilde{\varepsilon}_{1,t+1-i}^{(i)}(\delta_T^y)+\beta_1\tilde{\underline{u}}_{2,t-i}^{(i)}(\delta_T^y)-\hat{\beta}_1^{(i)}(\delta_T^y)\underline{u}_{2,t}\right]-\hat{\beta}_1^{(i)}(\delta_T^y)\left(\hat{\underline{u}}_{2,t}-\underline{u}_{2,t}\right)$$

$$= r_{1,t}^{(i)}(\delta_T^y)+r_{2,t}^{(i)}(\delta_T^y) \quad \text{for } i=0,1,2,3.$$

When $\delta_T^y=0$ and $i=0$ (A.34) further simplifies to

$$e_{t+1}\left(\hat{\beta}_1^{(0)},d_y\right) = \varepsilon_{1,t+1}+r_{2,t}^{(0)}+r_{3,t}^{(0)}. \quad \text{(A.35)}$$

The gradient (recall that we maximize only wrt d_y) is then written as

$$T^{-1/2}\bigtriangledown\widetilde{Q}_T(d_y) = T^{-1/2}\sum_{t=1}^{T-1}e_{t+1}\left(\hat{\beta}_1^{(0)},d_y\right)e_{t+1}^{(1)}\left(\hat{\beta}_1^{(1)},d_y\right) \quad \text{(A.36)}$$

$$= T^{-1/2}\sum_{t=1}^{T-1}e_{t+1}\left(\hat{\beta}_1^{(0)},d_y\right)r_{1,t}^{(1)}+o_p(1)=O_p(1)$$

by (A.28)-(A.33) and the Cauchy-Schwarz inequality and since $\alpha_x=1/2$. The requirement that $\alpha_x=1/2$ is needed to bound the summations involving $r_{2,t}^{(0)}r_{1,t}^{(1)}$ and $r_{3,t}^{(0)}r_{1,t}^{(1)}$. This establishes the approximation in (A.36), which with $B_T=T^{1/2}$, satisfies conditions (i) and (ii) of (Andrews and Sun 2004, Lemma 1).

Condition (iii) of (Andrews and Sun 2004, Lemma 1), requires, with probability approaching one, that

$$B_T^{-2}\bigtriangledown^2\widetilde{Q}_T(d_y)=T^{-1}\sum_{t=1}^{T-1}e_{t+1}^{(1)}\left(d_y,\hat{\beta}_1^{(1)}\right)^2+T^{-1}\sum_{t=1}^{T-1}e_{t+1}\left(d_y,\hat{\beta}_1^{(0)}\right)e_{t+1}^{(2)}\left(d_y,\hat{\beta}_1^{(2)}\right) \quad \text{(A.37)}$$

be bounded above zero. The second term on the RHS of (A.37) is $o_p(1)$ since, substituting (A.34) and (A.35),

$$T^{-1}\sum_{t=1}^{T-1} e_{t+1}\left(d_y, \hat{\beta}_1^{(0)}\right) e_{t+1}^{(2)}\left(d_y, \hat{\beta}_1^{(2)}\right) = T^{-1}\sum_{t=1}^{T-1}\left(\varepsilon_{1,t+1} + r_{2,t}^{(0)} + r_{3,t}^{(0)}\right)\times\left(r_{1,t}^{(2)} + r_{2,t}^{(2)}\right) = o_p(1) \tag{A.38}$$

by (A.28)-(A.33). Using (A.34) to substitute for $e_{t+1}^{(1)}\left(d_y, \hat{\beta}_1^{(1)}\right)^2$ by (A.28)-(A.29),

$$T^{-1}\sum_{t=1}^{T-1} e_{t+1}^{(1)}\left(d_y, \hat{\beta}_1^{(1)}\right)^2 = T^{-1}\sum_{t=1}^{T-1}\left(r_{1,t}^{(1)} + r_{2,t}^{(1)}\right)^2 = T^{-1}\sum_{t=1}^{T-1}\left(r_{1,t}^{(1)}\right)^2 + o_p(1). \tag{A.39}$$

Using[20] $\bar{\gamma}_{r_1,r_1}(0) = \lim_{T\to\infty} T^{-1}\sum_{t=1}^{T} E\left[r_{1,t}^2\right] = \lim_{T\to\infty} T^{-1}\sum_{t=1}^{T} E\left[\left(\tilde{\varepsilon}_{1,t}^{(1)} + \beta_1 \underline{\tilde{u}}_{2,t-1}^{(1)} - \beta_1^{(1)}\underline{u}_{2,t}\right)^2\right]$,

$$T^{-1}\sum_{t=1}^{T-1}\left(r_{1,t}^{(1)}\right)^2 = T^{-1}\sum_{t=1}^{T}\left[\left(\tilde{\varepsilon}_{1,t}^{(1)} + \beta_1 \underline{\tilde{u}}_{2,t-1}^{(1)} - \beta_1^{(1)}\underline{u}_{2,t}\right)^2\right] + o_p(1) = \bar{\gamma}_{r_1,r_1}(0) + o_p(1) > 0 + o_p(1) \tag{A.40}$$

using (A.25) to substitute for $r_{1,t}$ and where the convergence to $\bar{\gamma}_{r_1,r_1}(0)$ is established by applying (A.7) to each of the six constituent summands (i.e. $T^{-1}\sum_{t=1}^{T}(\tilde{\varepsilon}_{1,t}^{(1)})^2$, $T^{-1}\sum_{t=1}^{T}\tilde{\varepsilon}_{1,t}^{(1)}\underline{\tilde{u}}_{2,t-1}^{(1)}$, ..., $T^{-1}\sum_{t=}^{T}\underline{u}_{2,t}^2$). The inequality is strict since the possibility that $\tilde{\varepsilon}_{1,t}^{(1)} + \beta_1\underline{\tilde{u}}_{2,t-1}^{(1)} - \beta_1^{(1)}\underline{u}_{2,t} = 0$ a.s. is ruled out by (3) and (4). Combining (A.38)-(A.40),

$$B_T^{-2}\,\triangledown^2\,\widetilde{Q}_T(d_y) = \bar{\gamma}_{r_1,r_1}(0) + o_p(1) > 0 + o_p(1), \tag{A.41}$$

which establishes condition (iii) of (Andrews and Sun 2004, Lemma 1).

For (iv), let $K_T = ln(T) \to \infty$ as $T \to \infty$ (note that condition (iv) is required to hold only for *some* sequence $K_T \to \infty$) and let $\bar{\delta} > 0$ and $\epsilon > 0$ be selected such that $2\bar{\delta} + \epsilon < 1/2$. Then by an exact Taylor expansion of $\triangledown^2\widetilde{Q}_T(d_y + \delta_T^y)$ about d_y

$$\sup_{|\delta_T^y|<T^{-1/2}K_T} T^{-1}\left(\triangledown^2\widetilde{Q}_T(d_y+\delta_T^y) - \triangledown^2\widetilde{Q}_T(d_y)\right) = \sup_{|\delta_T^y|<T^{-1/2}K_T} T^{-(1+2\bar{\delta}+\epsilon)}\triangledown^3\widetilde{Q}_T(d_y+\delta_T^y)T^{2\bar{\delta}+\epsilon}\delta_T^{*y} \tag{A.42}$$

where δ_T^{*y} lies between 0 and δ_T^y. Then $\sup_{|\delta_T^y|<T^{-1/2}K_T} T^{2\bar{\delta}+\epsilon}|\delta_T^{*y}| \leq \sup_{|\delta_T^y|<T^{-1/2}K_T} T^{2\bar{\delta}+\epsilon}|\delta_T^y| = T^{2\bar{\delta}+\epsilon-1/2}K_T = o(1)$ by assumption. It remains to be shown that $\sup_{|\delta_T^y|<T^{-1/2}K_T} T^{-(1+2\bar{\delta}+\epsilon)}\,\triangledown^3\,\widetilde{Q}_T(d_y+\delta_T^y) = O_p(1)$. The restriction $|\delta_T^y| < T^{-1/2}K_t$ implies that $|\delta_T^y| < \bar{\delta}$ will be satisfied for T sufficiently large that $Tln(T)^{-2} > \bar{\delta}^{-2}$. Therefore it will be sufficient to show that $\sup_{|\delta_T^y|<\bar{\delta}} T^{-(1+2\bar{\delta}+\epsilon)}\,\triangledown^3\,\widetilde{Q}_T(d_y+\delta_T^y) = O_p(1)$. The third derivative in (A.42) is given by

$$\begin{aligned} B_T^{-2}\,\triangledown^3\,\widetilde{Q}_T(d_y+\delta_T^y) &= 3T^{-1}\sum_{t=1}^{T-1} e_{t+1}^{(1)}\left(d_y+\delta_T^y, \hat{\beta}_1^{(1)}(\delta_T^y)\right) e_{t+1}^{(2)}\left(d_y+\delta_T^y, \hat{\beta}_1^{(2)}(\delta_T^y)\right) \quad \text{(A.43)} \\ &+ T^{-1}\sum_{t=1}^{T-1} e_{t+1}\left(d_y+\delta_T^y, \hat{\beta}_1^{(0)}(\delta_T^y)\right) e_{t+1}^{(3)}\left(d_y+\delta_T^y, \hat{\beta}_1^{(3)}(\delta_T^y)\right). \end{aligned}$$

[20] Here and in the equation below, we use the consistency of $\hat{\beta}_1^{(1)}$ for $\beta_1^{(1)}$.

Using (A.34), the supremum of the top line of (A.43) is given by

$$\sup_{|\delta_T^y|<\bar{\delta}} 3T^{-(1+2\bar{\delta}+\epsilon)} \sum_{t=1}^{T} \left(r_{1,t}^{(1)}(\delta_T^y) + r_{2,t}^{(1)}(\delta_T^y)\right) \left(r_{1,t}^{(2)}(\delta_T^y) + r_{2,t}^{(2)}(\delta_T^y)\right) = O_p\left(T^{-\epsilon}\ln(T)\right) = o_p(1),$$

where the order follows from (A.28)-(A.29) and (A.31)-(A.32). Similarly, from (A.34) the supremum of the bottom line of (A.43) is given by

$$\sup_{|\delta_T^y|<\bar{\delta}} T^{-(1+2\bar{\delta}+\epsilon)} \sum_{t=1}^{T-1} \left(r_{1,t}^{(0)}(\delta_T^y) + r_{2,t}^{(0)}(\delta_T^y)\right) \left(r_{1,t}^{(3)}(\delta_T^y) + r_{2,t}^{(3)}(\delta_T^y)\right) = O_p\left(T^{-\epsilon}\ln(T)^2\right) = o_p(1),$$

where the order follows by (A.28)-(A.29). Therefore the RHS of (A.42) is $o_p(1)$, which establishes condition (iv) of (Andrews and Sun 2004, Lemma 1). The stated result then follows from the application of their Lemma.

Proof of Corollary 2.7

$\hat{d}_y \to_p d_y$ as an immediate consequence of Theorem 2.6. $(1-L)^{d_y} y_{t+1}$ in (10) is an I(0) dependent variable, analogous to y_{t+1} in (1). Thus as a consequence of Theorem 2.1 (here $(1-L)^{d_y} y_{t+1}$ replaces y_{t+1} in the proof) $\hat{\beta}_1(d_y, \hat{d}_x) \to_p \beta_1$. Then since $\hat{d}_y \to_p d_y$, by the continuous mapping theorem $\hat{\beta}_1(\hat{d}_y, \hat{d}_x) \to_p \beta_1$.

Proof of Theorem 2.8

Define $\hat{\delta}_T^y = \hat{d}_y - d_y$ and let $\bar{\delta} > 0$. Due to the consistency of both $\hat{d}_y$ and $\hat{d}_x$, we may assume $|\hat{\delta}_T^y|, |\hat{\delta}_T^x| < \bar{\delta}$ without loss of generality using the same arguments given below (A.13). Similarly to (A.15)

$$\begin{aligned} (1-L)^{\hat{\delta}_T^y} u_{2,t} 1_{\{t>0\}} &= u_{2,t} 1_{\{t>0\}} + \hat{\delta}_T^y \tilde{u}_{2,t-1}^{(1)} + \frac{1}{2}(\hat{\delta}_T^y)^2 \tilde{u}_{2,t-2}^{(2)}(\delta_T^*) \quad \text{and} \\ (1-L)^{\hat{\delta}_T^y} \varepsilon_{1,t+1} 1_{\{t>0\}} &= \varepsilon_{1,t+1} 1_{\{t>0\}} + \hat{\delta}_T^y \tilde{\varepsilon}_{1,t}^{(1)} + \frac{1}{2}(\hat{\delta}_T^y)^2 \tilde{\varepsilon}_{1,t-1}^{(2)}(\delta_T^*) \end{aligned} \tag{A.44}$$

using the definitions in Lemmas A.1 and A.2 where δ_T^* lies between 0 and $\hat{\delta}_T^y$. By Theorem 2.6

$$\begin{aligned} T^{1/2}\hat{\delta}_T^y &= -\left(T^{-1} \bigtriangledown^2 \widetilde{Q}_T(d_y)\right)^{-1} T^{-1/2} \bigtriangledown \widetilde{Q}_T(d_y) \\ &= -\bar{\gamma}_{r_1^{(1)}, r_1^{(1)}}(0)^{-1} T^{-1/2} \sum_{t=1}^{T-1} e_{t+1}\left(\hat{\beta}_1^{(0)}, d_y\right) r_{1,t}^{(1)} + o_p(1) = O_p(1), \end{aligned} \tag{A.45}$$

where $r_{1,t}^{(i)}$ is defined in (A.25) and using the definitions of Lemma A.2. The final result in (A.45) follows by (A.36), and (A.41). Note that when $\beta_1 = 0$, $\bar{\gamma}_{r_1^{(1)}, r_1^{(1)}}(0) = \bar{\gamma}_{\tilde{\varepsilon}_1^{(1)}, \tilde{\varepsilon}_1^{(1)}}(0) - 2\beta_1^{(1)} \bar{\gamma}_{u_2, \tilde{\varepsilon}_1^{(1)}}(0) + (\beta_1^{(1)})^2 \gamma_{u_2, u_2}(0)$.

From (18), using (10) to substitute for y_{t+1}, we have (suppressing the indicator function)

$$\hat{\beta}_1(\hat{d}_y) = \left(\sum_{t=1}^{T-1} \underline{\hat{u}}_{2,t}^2\right)^{-1} \sum_{t=1}^{T-1} \underline{\hat{u}}_{2,t}(1-L)^{\hat{d}_y} y_{t+1} = \left(\sum_{t=1}^{T-1} \underline{\hat{u}}_{2,t}^2\right)^{-1} \sum_{t=1}^{T-1} \underline{\hat{u}}_{2,t}(1-L)^{\hat{\delta}_T^y}\left([\beta_1 u_{2,t} + \varepsilon_{1,t+1}]\right)$$

$$= \beta_1 + \left(\sum_{t=1}^{T-1} \underline{\hat{u}}_{2,t}^2\right)^{-1} \sum_{t=1}^{T-1} \left[\beta_1 \underline{\hat{u}}_{2,t}\left((1-L)^{\hat{\delta}_T^y} u_{2,t} - \underline{\hat{u}}_{2,t}\right) + \underline{\hat{u}}_{2,t}(1-L)^{\hat{\delta}_T^y}\varepsilon_{1,t+1}\right]. \quad \text{(A.46)}$$

Similarly to (A.13), (A.46) can be rearranged as

$$\sqrt{T}(\hat{\beta}_1(\hat{d}_y) - \beta_1) = \left(T^{-1}\sum_{t=1}^{T-1} \underline{\hat{u}}_{2,t}^2\right)^{-1} \left(T^{-1/2}\sum_{t=1}^{T-1} \underline{\hat{u}}_{2,t}(1-L)^{\hat{\delta}_T^y}\varepsilon_{1,t+1}\right. \quad \text{(A.47)}$$

$$\left. + \beta_1 T^{-1/2}\sum_{t=1}^{T-1} \underline{\hat{u}}_{2,t}\left[(1-L)^{\hat{\delta}_T^y} u_{2,t} - \underline{\hat{u}}_{2,t}\right]\right).$$

The denominator of (A.47) is given by (A.18). For the second term in the numerator of (A.47), using (A.15) and (A.44) to substitute for $\hat{u}_{2,t}$ and $(1-L)^{\hat{\delta}_T^y}\varepsilon_{1,t+1}$ respectively,

$$T^{-1/2}\sum_{t=1}^{T-1} \underline{\hat{u}}_{2,t}(1-L)^{\hat{\delta}_T^y}\varepsilon_{1,t+1} = T^{-1/2}\sum_{t=1}^{T-1}\left(u_{2,t} + \hat{\delta}_T^x \tilde{u}_{2,t-1}^{(1)} + \frac{1}{2}(\hat{\delta}_T^x)^2 \tilde{u}_{2,t-2}^{(2)}(\delta_T^*)\right) \quad \text{(A.48)}$$

$$\times\left(\varepsilon_{1,t+1} + \hat{\delta}_T^y \tilde{\varepsilon}_{1,t}^{(1)} + \frac{1}{2}(\hat{\delta}_T^y)^2 \tilde{\varepsilon}_{1,t-1}^{(2)}(\delta_T^*)\right)$$

$$= T^{-1/2}\sum_{t=1}^{T-1} u_{2,t}\left(\varepsilon_{1,t+1} + \hat{\delta}_T^y \tilde{\varepsilon}_{1,t}^{(1)}\right) + o_p(1),$$

where $T^{-1/2}\sum_{t=1}^{T-1}\left(\hat{\delta}_T^x \tilde{u}_{2,t-1}^{(1)} + \frac{1}{2}(\hat{\delta}_T^x)^2 \tilde{u}_{2,t-2}^{(2)}(\delta_T^*)\right)\varepsilon_{1,t+1} = o_p(1)$, by (A.17), and the remaining orders follow by Lemma A.2, using the Cauchy-Schwarz and Hölder inequalities, since $\hat{\delta}_T^y = O_p(T^{-1/2})$ by (A.45) and $\hat{\delta}_T^x = O_p(T^{-1/2})$ by assumption. Using Lemma A.2, (A.48) further simplifies to

$$T^{-1/2}\sum_{t=1}^{T-1} \underline{\hat{u}}_{2,t}(1-L)^{\hat{\delta}_T^y}\varepsilon_{1,t+1} = T^{-1/2}\sum_{t=1}^{T-1}\varepsilon_{1,t+1}u_{2,t} + \bar{\gamma}_{u_2,\tilde{\varepsilon}_1^{(1)}}(0)T^{1/2}\hat{\delta}_T^y + o_p(1). \quad \text{(A.49)}$$

Next, use (A.45) to substitute for $\hat{\delta}_T^y$ and re-arrange to obtain

$$T^{-1/2}\sum_{t=1}^{T-1} \underline{\hat{u}}_{2,t}(1-L)^{\hat{\delta}_T^y}\varepsilon_{1,t+1} = T^{-1/2}\sum_{t=1}^{T-1}\left(\varepsilon_{1,t+1}u_{2,t} - \bar{\gamma}_{u_2,\tilde{\varepsilon}_1^{(1)}}(0)(\bar{\gamma}_{r_1^{(1)},r_1^{(1)}}(0))^{-1} e_{t+1}\left(\hat{\beta}_1^{(0)}, d_y\right) r_{1,t}^{(1)}\right)$$

$$+ o_p(1).$$

Using (A.35) to substitute for $e_{t+1}\left(\hat{\beta}_1^{(0)}, d_y\right) = \varepsilon_{1,t+1} + r_{2,t}^{(0)} + r_{3,t}^{(0)}$ and further re-arranging gives

$$T^{-1/2}\sum_{t=1}^{T-1} \underline{\hat{u}}_{2,t}(1-L)^{\hat{\delta}_T^y}\varepsilon_{1,t+1} = T^{-1/2}\sum_{t=1}^{T-1}\varepsilon_{1,t+1}\left(u_{2,t} - \bar{\gamma}_{u_2,\tilde{\varepsilon}_1^{(1)}}(0)(\bar{\gamma}_{r_1^{(1)},r_1^{(1)}}(0))^{-1} r_{1,t}^{(1)}\right) \quad \text{(A.50)}$$

$$-\bar{\gamma}_{u_2,\tilde{\varepsilon}_1^{(1)}}(0)(\gamma_{r_1^{(1)},r_1^{(1)}}(0))^{-1} T^{-1/2}\sum_{t=1}^{T-1}\left(r_{2,t}^{(0)} + r_{3,t}^{(0)}\right) r_{1,t}^{(1)}.$$

Note that from the definition in (A.26) we have, using Theorem 2.1,[21] Lemma A.2, (A.15), and (A.28)-(A.29),

$$-T^{-1/2}\sum_{t=1}^{T-1} r_{2,t}^{(0)} r_{1,t}^{(1)} = \beta_1 T^{-1/2}\sum_{t=1}^{T-1} r_{1,t}^{(1)}(\hat{u}_{2,t}-u_{2,t})+o_p(1) \quad \text{(A.51)}$$
$$= \beta_1 T^{-1/2}\sum_{t=1}^{T-1} r_{1,t}^{(1)}\left(\hat{\delta}_T^x \tilde{u}_{2,t-1}^{(1)}+\frac{1}{2}(\hat{\delta}_T^x)^2\tilde{u}_{2,t-2}^{(2)}(\delta_T^*)\right)+o_p(1)=\beta_1 O_p(1)+o_p(1).$$

Noting the definitions in (A.25) and (A.27) and using (9) (recall here $\alpha_x = 1/2$),

$$-T^{-1/2}\sum_{t=1}^{T-1} r_{3,t}^{(0)} r_{1,t}^{(1)} = \left(\hat{\beta}_1^{(0)}-\beta_1\right)T^{-1/2}\sum_{t=1}^{T-1} u_{2,t}\left[\tilde{\varepsilon}_{1,t}^{(1)}+\beta_1\tilde{u}_{2,t-1}^{(1)}-\hat{\beta}_1^{(1)}u_{2,t}\right] \quad \text{(A.52)}$$
$$= T^{1/2}\left(\hat{\beta}_1^{(0)}-\beta_1\right)\left[\bar{\gamma}_{u_2,\tilde{\varepsilon}_1^{(1)}}(0)-\beta_1^{(1)}\gamma_{u_2,u_2}(0)\right]+\beta_1 O_p(1)+o_p(1)$$
$$= \beta_1 O_p(1)+o_p(1)$$

since $\beta_1^{(1)}=\gamma_{u_2,u_2}(0)^{-1}\left(\beta_1\bar{\gamma}_{\tilde{u}_2^{(1)},u_2}(1)+\bar{\gamma}_{u_2,\tilde{\varepsilon}_1^{(1)}}(0)\right)$ (see the discussion above (A.25)).

Therefore when $\beta_1=0$, the limiting distribution of the numerator is only affected by the top line of (A.50), for which

$$T^{-1/2}\sum_{t=1}^{T-1}\varepsilon_{1,t+1}\left(u_{2,t}-\bar{\gamma}_{u_2,\tilde{\varepsilon}_1^{(1)}}(0)(\bar{\gamma}_{r_1^{(1)},r_1^{(1)}}(0))^{-1}r_{1,t}^{(1)}\right) = T^{-1/2}\sum_{t=1}^{T-1}\varepsilon_{1,t+1}\xi_t+o_p(1) \quad \text{(A.53)}$$
$$\to_d N\left(0,\bar{\gamma}_{\xi,\xi}(0)\Sigma_{1,1}\right)$$

where we define $\xi_t = u_{2,t}-\bar{\gamma}_{u_2,\tilde{\varepsilon}_1^{(1)}}(0)\bar{\gamma}_{r_1^{(1)},r_1^{(1)}}(0)^{-1}r_{1,t}^{(1)}$. Next, by (Gradstein and Ryzhik 1994, eqn. 0.121) and Toeplitz Lemma we may reexpress

$$\bar{\gamma}_{\tilde{\varepsilon}_1^{(1)},\tilde{\varepsilon}_1^{(1)}}(0)=\Sigma_{1,1}\frac{\pi^2}{6}. \quad \text{(A.54)}$$

Noting that when $\beta_1=0$, $r_{1,t}^{(1)}=\tilde{\varepsilon}_{1,t}^{(1)}-\hat{\beta}_1^{(1)}\underline{u}_{2,t}$ and $\beta_1^{(1)}=\beta_1+o_p(1)=\gamma_{u_2,u_2}(0)^{-1}\bar{\gamma}_{u_2,\tilde{\varepsilon}_1^{(1)}}(0)+o_p(1)$,

$$\bar{\gamma}_{u_2,r_1^{(1)}} = \bar{\gamma}_{u_2,\varepsilon_1^{(1)}}(0)-\beta_1^{(1)}\gamma_{u_2,u_2}(0)=0 \quad \text{and} \quad \text{(A.55)}$$

$$\bar{\gamma}_{r_1,r_1}(0) = \bar{\gamma}_{\tilde{\varepsilon}_1^{(1)},\tilde{\varepsilon}_1^{(1)}}(0)-2\beta_1^{(1)}\bar{\gamma}_{u_2,\tilde{\varepsilon}_1^{(1)}}(0)+(\beta_1^{(1)})^2\gamma_{u_2,u_2}(0) \quad \text{(A.56)}$$
$$= \bar{\gamma}_{\tilde{\varepsilon}_1^{(1)},\tilde{\varepsilon}_1^{(1)}}(0)-\gamma_{u_2,u_2}(0)^{-1}\bar{\gamma}_{u_2,\tilde{\varepsilon}_1^{(1)}}(0)^2$$

implying that

$$\bar{\gamma}_{\xi,\xi}(0) = lim_{T\to\infty}T^{-1}\sum_{t=1}^{T}E\left[\xi_t^2\right]=\gamma_{u_2,u_2}(0)+\bar{\gamma}_{r_1^{(1)},r_1^{(1)}}(0)^{-1}\bar{\gamma}_{u_2,\tilde{\varepsilon}_1^{(1)}}(0)^2 \quad \text{(A.57)}$$
$$= \gamma_{u_2,u_2}(0)+\left(\Sigma_{1,1}\frac{\pi^2}{6}\bar{\gamma}_{u_2,\tilde{\varepsilon}_1^{(1)}}(0)^{-2}-\gamma_{u_2,u_2}(0)^{-1}\right)^{-1}.$$

[21]Since $\hat{\beta}_1$ is an implicit function of the true value of d_y the consistency result from Theorem 2.1 implies that $\hat{\beta}_1=\beta_1+o_p(1)$.

Since $E_t\varepsilon_{1,t+1} = 0$ and ξ_t is realized at time t, $\varepsilon_{1,t+1}\xi_t$ is a MDS. The central limit theorem therefore follows by Davidson (2000, Theorem 6.2.3, p. 124), giving $\sqrt{T}\left(\hat{\beta}_1(\hat{d}_y) - \beta_1\right) \rightarrow_d N(0, V)$ where $V = \gamma_{u_2,u_2}(0)^{-2}\bar{\gamma}_{\xi,\xi}(0)\Sigma_{1,1} = \Sigma_{1,1}\left[\gamma_{u_2,u_2}(0) - \frac{6}{\pi^2}\Sigma_{1,1}^{-1}\bar{\gamma}_{u_2,\tilde{\varepsilon}_1^{(1)}}(0)^2\right]^{-1}$.

Proof of Corollary 2.9

The stated results follows from Theorem 2.8 after establishing the convergence of the sample analogs to their population counterparts. In what follows below, we provide the argument for the convergence of $T^{-1}\sum_{t=1}^{T-1}(\hat{\varepsilon}_{1,t}^{(1)})^2$ to $\bar{\gamma}_{\tilde{\varepsilon}_1^{(1)},\tilde{\varepsilon}_1^{(1)}}(0)$. The convergence of the remaining terms follow by similar argument and are omitted to conserve space. Using (10) to substitute for $(1-L)^{d_y}y_{t+1}$ under $H_0 : \beta_1 = 0$, we have

$$\hat{\varepsilon}_{1,t+1} = \tilde{\varepsilon}_{1,t+1}^{(1)}(\hat{\delta}_T^y) - \hat{\beta}_1 \ln(l - L)\hat{u}_{2,t},$$

where $\hat{\delta}_T^y = \hat{d}_y - d_y$ and $\tilde{\varepsilon}_{1,t+1}^{(1)}$ is defined as in Lemma A.1. Applying $\ln(1 - L)$ to both sides of an expansion analogous to (A.14), we obtain

$$\tilde{\varepsilon}_{1,t+1}^{(1)}(\hat{\delta}_T^y) = \tilde{\varepsilon}_{1,t}^{(1)} + \hat{\delta}_T^y\tilde{\varepsilon}_{1,t-1}^{(2)} + 1/2(\hat{\delta}_T^y)^2\tilde{\varepsilon}_{1,t-2}^{(3)}(\delta_T^*),$$

with $|\delta_T^*| < \hat{\delta}_T^y$. By (A.7) $T^{-1}\sum_{t=1}^{T-1}(\tilde{\varepsilon}_{1,t+1}^{(1)})^2 \rightarrow_p \bar{\gamma}_{\tilde{\varepsilon}_1^{(1)},\tilde{\varepsilon}_1^{(1)}}(0)$. It remains to show that the other terms are of lower order. Noting that $\hat{\delta}_T^y = O_p(T^{-1/2})$ by (A.45), it follows from rate results of Lemma A.2 that

$$T^{-1}\sum_{t=1}^{T-1}\tilde{\varepsilon}_{1,t+1}^{(1)}(\hat{\delta}_T^y)^2 = T^{-1}\sum_{t=1}^{T-1}(\tilde{\varepsilon}_{1,t+1}^{(1)})^2 + o_p(1).$$

Next, applying $\ln(1 - L)$ to (A.14)

$$\ln(1 - L)\hat{u}_{2,t} = \tilde{u}_{2,t-1}^{(1)} + \hat{\delta}_T^x\tilde{u}_{2,t-2}^{(2)} + 1/2(\hat{\delta}_T^x)^2\tilde{u}_{2,t-3}^{(3)}(\delta_T^*),$$

with $|\delta_T^x| < \hat{\delta}_T^x$. Then, since $\hat{\beta}_1 \rightarrow_p \beta_1 = 0$ and $\hat{\delta}_T^x = O_p(T^{-1/2})$ by assumption, and again using the results of Lemma A.2, it is straightforward to show that

$$T^{-1}\sum_{t=1}^{T-1}\hat{\varepsilon}_t^2 = T^{-1}\sum_{t=1}^{T-1}\tilde{\varepsilon}_{1,t+1}^{(1)}(\hat{\delta}_T^y)^2 + o_p(1).$$

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